EIGENFUNCTION EXPANSIONS ASSOCIATED WITH THE LAPLACIAN FOR CERTAIN DOMAINS WITH INFINITE BOUNDARIES. III(1)

To the memory of my father, Emil Goldstein

BY
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1. **Introduction.** In the present paper we shall consider the effect of small perturbations on the spectrum of the selfadjoint operator $-\Delta$ associated with certain homogeneous boundary conditions in a domain S. We shall then apply these results to scattering theory. In all of the cases considered, a spectral representation for the unperturbed operator is obtained with the aid of separation of variables.

In §§2–4, S will represent a wedge shaped region in two dimensional Euclidean space (R^2), and the unperturbed operator A_0 will be given by $-\Delta$ acting on functions which vanish on $\dot{S}$ (the boundary of S). (Higher dimensional cones may be treated similarly.) We shall first perturb the operator A_0 by altering a finite portion of the boundary of S. We assume that the perturbed domain Ω has a sufficiently smooth boundary $\dot{\Omega}$. The new operator A is defined in the same manner as A_0 with S replaced by Ω.

We shall prove that A is unitarily equivalent to A_0 by employing the method of distorted plane waves, first used to prove an expansion theorem by Ikebe [7]. This method was also used to deal with the exterior problem for the Laplacian by Shenk [11], and Shizuta [13]. The author [5](2), extended this method to treat certain domains with infinite boundaries. The domains in I were perturbed infinite cylinders.

In §2, we consider the unperturbed operator A_0. A complete, orthonormal set of generalized eigenfunctions W_n^0 are exhibited. Employing the functions W_n^0, we immediately obtain a spectral representation for A_0. The spectrum of A_0 is absolutely continuous and the spectral multiplicity is infinite at each point in the spectrum of A_0. This differs from the case of an infinite cylinder, for which it was proven in I that the spectral multiplicity is nonuniform and finite throughout the spectrum. (Note that it follows from the results of §2 that the operators associated with different wedges are unitarily equivalent to each other.)

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(2) [5] and [6] will henceforth be referred to as I and II, respectively.

In §3, we construct two sets of generalized eigenfunctions $W_n^{\pm}$ of the operator A. The method used is that of limiting absorption, employed by Eidus [4], to solve a boundary value problem in a perturbed semi-infinite cylinder. The functions $W_n^+ - W_n^0(W_n^- - W_n^0)$ will satisfy a certain set of "outgoing" ("incoming") radiation conditions at infinity.

In §4, the functions W_n^+ are shown to form a complete orthonormal set of generalized eigenfunctions. The proofs follow along the same lines as those in I. The major difference occurs in the statement and proof of Lemma 4.2. This is due to the fact that in I, we assumed that $\Omega \subset S$, whereas, using the arguments of this section, we need not impose this restriction. There are also additional technical difficulties due to the fact that the spectrum of A has infinite multiplicity. Observe that the results of this section include the case in which S is the entire plane. However, we have used a different set of generalized eigenfunctions than those employed in [11] and [13], ours being countable in number for each $\lambda > 0$, whereas the eigenfunctions in [11] and [13] are uncountable.

In §5, we apply our results to scattering theory. In particular, we establish the existence, unitarity, and invariance of the wave operators

$$W_{\pm}(\phi(A^{1/2}), \phi(A_0^{1/2})) = \lim_{t \to \pm\infty} \exp[it\phi(A^{1/2})]J \exp[-it\phi(A_0^{1/2})]f,$$

where ϕ belongs to a wide class of real-valued functions and J denotes a continuous mapping from $L_2(S)$ into $L_2(\Omega)$. Analogous results were obtained by Kato [9], Shenk [12], and the author, II. We apply our results to obtain the existence and unitarity of the scattering operator $\mathscr{S} = (W^+)^{-1}W^-$. We extend these results to the wave equation in Ω.

In §6, we express the operator $\mathscr{S}$ as a matrix whose elements depend explicitly on the outgoing waves. In §7, the previous results are extended in several directions. Different domains (of arbitrary dimension) are considered as well as more general boundary conditions. Also, the operator A_0 may be perturbed by varying the coefficients (for example by adding a potential to $-\Delta$) or by perturbing the boundary conditions. The same methods are applicable to these problems.

We remark at this point that results of Birman [2], imply the unitary equivalence of the absolutely continuous parts of A and A_0. No mention is made in [2] however, of the outgoing or incoming waves.

2. **Preliminaries.** The main result of this section is an analogue of the Fourier inversion formula. This will follow from Theorem 2.1. We begin with some definitions. Let $X = (X_1, X_2)$ represent an arbitrary point in R^2 and set $r = (X_1^2 + X_2^2)^{1/2}$, $\theta = \tan^{-1}(X_2/X_1)$. Let S denote the cone: $r \geqq 0$, $\pi\alpha \geqq \theta \geqq 0$, where $2 \geqq \alpha > 0$. We perturb S to obtain a domain Ω satisfying the following conditions:

(1) $\Omega = S$ for $r \geqq r_0$.

(2) $\dot{\Omega}$ is a C^2 curve with the possible exception of the point $r = 0$. If $r = 0$ is a singular point then it is bounded away from $\dot{\Omega} - \dot{S}$.

(3) $\Omega - S$ is made up of at most a finite number of domains D_j such that each D_j is obtained by reflecting a bounded subset $D_j^* \subset S$ successively across the lines $\theta = \pi\alpha$, $\theta = 2\pi\alpha, \ldots, \theta = M\pi\alpha$ ($M\alpha < 2$).

We denote by Ω_{r_1} ($\dot{\Omega}'_{r_1}$) the set of points X in Ω ($\dot{\Omega}$) for which $r \leqq r_1$, and let l_{r_1} represent the intersection of Ω with the circle $r = r_1$. Clearly, we have $\dot{\Omega}_{r_1} = \dot{\Omega}'_{r_1} \cup l_{r_1}$. We define Ω_{r_1,r_2} ($\dot{\Omega}'_{r_1,r_2}$) as the set of points X in Ω ($\dot{\Omega}$) for which $r_1 \leqq r \leqq r_2$.

Let us recall the notation used in I. Suppose B is an arbitrary domain in R^1 or R^2, m is a nonnegative integer, and $\phi(x) \in C_0^\infty(B)$. Set

$$\|\phi\|_{m(B)} = \left(\sum_{|\alpha| \leqq m} \int_B |D^\alpha \phi(x)|^2 \, dx \right)^{1/2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and each α_j is a nonnegative integer. Denote by $\dot{H}_m(B)$ ($H_m(B)$), the completion of $C_0^\infty(B)$ ($C^\infty(B)$) under the norm given by $\| \quad \|_{m(B)}$. Let $H_m^{\mathrm{loc}}(B)$ consist of those functions $f(X)$ satisfying the condition $f(X) \in H_m(B')$ for every bounded subset $B' \subset B$. Finally, set $\|\mu\|_{m(\Omega_r)} = \|\mu\|_{m(r)}$ and $\|\mu\|_{m(\Omega_{r_1,r_2})} = \|\mu\|_{m(r_1,r_2)}$.

We next define a selfadjoint operator A_B acting in the Hilbert space $L_2(B)$. A_B will denote the Friedrichs extension of the operator given by $-\Delta$ acting on $C_0^\infty(B)$. It is well known that A_B is a nonnegative operator. Set $A_S = A_0$ and $A_\Omega = A$. In this section we consider the unperturbed operator A_0. We begin by constructing a complete, orthogonal set of generalized eigenfunctions $\dot{W}_n^0(X; \lambda)$ of A_0.

Suppose $\lambda > 0$. In polar coordinates, the equation $(\Delta + \lambda)\mu = 0$ becomes $\mu_{rr} + \mu_r/r + \mu_{\theta\theta}/r^2 + \lambda\mu = 0$. The boundary condition is $\mu(r, 0) = \mu(r, \pi\alpha) = 0$ for $r > 0$. Separating variables, we obtain the following solutions:

$$W_n^0(x; \lambda) = (\pi\alpha)^{-1/2} J_{n/\alpha}((\lambda)^{1/2} r) \sin(n\theta/\alpha),$$

where $J_{n/\alpha}$ denotes the Bessel function.

We define a Hilbert space H as follows. The elements of H will be sequences of functions $h = (h_1(\lambda), h_2(\lambda), \ldots) = \{h_n(\lambda)\}$ defined on $(0, \infty)$ and such that

$$\sum_{n=1}^\infty \int_0^\infty |h_n(\lambda)|^2 \, d\lambda < \infty.$$

Suppose $g = \{g_n(\lambda)\}$ and $h = \{h_n(\lambda)\} \in H$. Set

$$(g, h)_H = \sum_{n=1}^\infty \int_0^\infty g_n(\lambda) \overline{h_n(\lambda)} \, d\lambda.$$

Under the inner product given by $(\ , \ )_H$, H clearly forms a Hilbert space.

We next construct a pair of linear mappings from $L_2(S)$ into H and from H into $L_2(S)$. For each $\phi(X)$ in $C_0^\infty(S)$, set

$$T_0 \phi = \{\hat{\phi}_n^0(\lambda)\} = \left\{ \int_S \phi(X) W_n^0(X; \lambda) \, dx \right\}.$$

For each $\psi = \{\psi_n(\lambda)\}$ in H satisfying the conditions:

(2.1) $$\psi_n(\lambda) \in C_0^\infty(0, \infty),$$

and

$$\psi_n(\lambda) \equiv 0 \quad \text{for } n > M, \tag{2.2}$$

set

$$T_1\psi = \sum_{n=1}^{\infty} \int_0^{\infty} \psi_n(\lambda) W_n^0(X; \lambda)\, d\lambda.$$

The set of elements in H satisfying conditions (2.1) and (2.2) is dense in H.

THEOREM 2.1. *The transformations T_0 and T_1 may be extended to unitary transformations from $L_2(S)$ onto H and from H onto $L_2(S)$, respectively. Furthermore $T_0^* = T_0^{-1} = T_1$.*

The proof of Theorem 2.1 may be carried out in the usual manner and will be omitted. It is based on the following well-known result, [3]:

$$\int_0^{\infty} J_n(\xi r')\xi\, d\xi \int_0^{\infty} f(r) J_n(\xi r) r\, dr = f(r'),$$

where $f(r)$ is continuous in $(0, \infty)$ and $\int_0^\infty |f(r)| r^{1/2}\, dr < \infty$. It now follows from Theorem 2.1 that the functions $W_n^0(X; \lambda)$ form a complete, orthonormal set of generalized eigenfunctions. Note that for each function $\phi(X)$ in $D(A_0)$, (the domain of A_0), we have $T_0(A_0\phi) = \{\lambda\hat{\phi}_n^0(\lambda)\}$. Thus it follows from Theorem 2.1 that the mapping T_0 gives a spectral representation for the selfadjoint operator A_0. The spectrum of A_0, $\sigma(A_0)$, is $[0, \infty)$. $\sigma(A_0)$ is absolutely continuous and the spectral multiplicity $m_0(\lambda)$ of each λ in $\sigma(A_0)$ is ∞. We proceed to establish a theorem analogous to Theorem 2.1 for the operator A.

3. **The generalized eigenfunctions.** We shall see later in this section how the construction of a set of generalized eigenfunctions $W_n^+(X; \lambda)$ of the operator A reduces to the solution of the following boundary value problem. Given $F(X)$ in $C^\infty(\bar{\Omega})$, $F(X) \equiv 0$ for $r \geqq r_0$, $\lambda > 0$, find a function $\mu(X)$ such that

$$(-\Delta - \lambda)\mu = F. \quad \mu = 0 \text{ on } \dot{\Omega}. \tag{3.1}$$

Since λ is in the spectrum of A, [8], (3.1) is not a well-posed problem in $L_2(\Omega)$ for all such functions $F(X)$.

For the sake of simplicity, we shall carry out the proofs in the remainder of this paper for the case $\Omega = S \cup D$, where D^* (the reflection of D across the line $\theta = \pi\alpha$) is contained in S. The general domain Ω described in §2 may be treated simply by combining the method used here with that of I. For each point X in D, we denote by X^*, its reflection across $\theta = \pi\alpha$. We extend the definition of $W_n^0(X; \lambda)$ by reflecting as an odd function across $\theta = \pi\alpha$. Thus $W_n^0(X; \lambda) = -W_n^0(X^*; \lambda)$, $X \in D$. We denote the extended function again by $W_n^0(X; \lambda)$.

We shall solve (3.1) using the method of limiting absorption, [4]. Setting $\mu_\varepsilon = (A - \lambda - i\varepsilon)^{-1}F$ for $\varepsilon > 0$, we shall show as in I that $\mu_\varepsilon \rightarrow_\chi \mu$, where $\rightarrow_\chi$ denotes convergence in a topological space χ to be defined presently. The function $\mu(X)$

will be shown to be a solution of (3.1) which satisfies certain "outgoing" radiation conditions at infinity. We begin by obtaining a Fourier expansion for the function

$$\mu(X;\kappa) = (A-\kappa)^{-1}F(X), \quad \text{where } \operatorname{Im}\kappa > 0. \tag{3.2}$$

Set

$$d_n(r;\kappa) = \int_0^{\pi\alpha} \mu(r,\theta,\kappa)\sin\frac{n\theta}{\alpha}\,d\theta, \qquad n = 1, 2, \ldots.$$

Lemma 3.1. *For $r \geqq r_0$, we have*

$$\begin{aligned} d_n(r;\kappa) &= C_n(\kappa)H_n^{(1)}(\kappa^{1/2}r), \\ d_n'(r;\kappa) &= C_n(\kappa)(dH_n^{(1)}/dr)(\kappa^{1/2}r), \end{aligned} \tag{3.3}$$

where the $C_n(\kappa)$ are constants, $\operatorname{Im}\kappa^{1/2} > 0$, *and*

$$H_{n/\alpha}^{(1)} = J_{n/\alpha} + iY_{n/\alpha}, \tag{3.4}$$

$Y_{n/\alpha}$ denoting the Hankel function.

Proof. It follows readily from the properties of μ that $d_n(r;\kappa)$ satisfies Bessel's equation for $r \geqq r_0$. (3.3) now follows from the fact that $\mu \in L_2(\Omega)$. Q.E.D.

Remark 3.1. It is immediate that A can have no positive point eigenvalues, since the equation $A\mu = \lambda\mu$ $(\lambda > 0)$ implies as in Lemma 3.1 that for $r \geqq r_0$:

$$\mu(r,\theta) = \sum_{n=1}^{\infty} [C_n^{(1)}J_{n/\alpha}(\lambda^{1/2}r) + C_n^{(2)}Y_{n/\alpha}(\lambda^{1/2}r)]\sin\frac{n}{\alpha}\theta.$$

Furthermore, 0 cannot be a point eigenvalue of A, for suppose $A\mu = 0$. Then using the divergence theorem and the boundary condition $\mu = 0$ on $\dot{\Omega}$, we have

$$0 = \int_\Omega \mu(X)\,\overline{\Delta\mu(X)}\,dX = \int_\Omega |\nabla\mu(X)|^2\,dx.$$

Thus $\mu(X)$ is constant in Ω. Since $\mu(X) \in L_2(\Omega)$, we conclude that $\mu(X) \equiv 0$. Hence $[0,\infty)$ belongs to the continuous spectrum of A. Since A is a nonnegative self-adjoint operator, we see that A has a purely continuous spectrum.

We define the topological space χ as follows. χ will consist of those functions $\mu(X)$ satisfying the following conditions: (a) For each $\varepsilon > 0$, $r > \varepsilon$, $\mu(X) \in H_2(\Omega_{\varepsilon,r})$, (b) There exists a sequence of functions $\mu_n(X)$ such that for each $\varepsilon > 0$, $r > \varepsilon$, $\mu_n(X) \in H_2(\Omega_{\varepsilon,r})$, $\|\mu_n - \mu\|_{2(\varepsilon,r)} \to 0$ as $n \to \infty$, and each $\mu_n(X)$ vanishes in a neighborhood of $\dot{\Omega}$ (not necessarily the same neighborhood for each n). Given a sequence of functions $\{\mu_n\}$ in χ, we say that $\mu_n(X)$ converges to $\mu(X)$ in χ as $n \to \infty$ $(\mu_n \to_\chi \mu)$ if $\|\mu_n - \mu\|_{2(\varepsilon,r)} \to 0$ for each $\varepsilon > 0$, $r > \varepsilon$. Finally, for $\mu(X)$ in χ, set $-\Delta\mu = \bar{A}\mu$.

Theorem 3.1 (Uniqueness theorem). *Suppose $\mu(X) \in H_1(\Omega_{\varepsilon,r_1})$ (for a fixed $r_1 > r_0$, and each $0 < \varepsilon < r_1$), $\mu(X) = 0$ on $\dot{\Omega}'_{r_1}$, and $-\Delta\mu = \lambda\mu$ $(\lambda > 0)$ in Ω_{r_1}. Furthermore, suppose*

$$d_n(r) = \int_{l_r} \mu(r,\theta)\sin\frac{n\theta}{\sigma}\,d\theta = C_n(\lambda)H_{n/\alpha}^{(1)}(\lambda^{1/2}r), \tag{3.5}$$

and

$$d_n'(r) = \int_{l_r} \frac{\partial}{\partial r}\mu(r,\theta)\sin\frac{n\theta}{\sigma}\,d\theta = C_n(\lambda)\frac{d}{dr}H^{(1)}_{n/\alpha}(\lambda^{1/2}r) \tag{3.6}$$

for $r_1 \geqq r \geqq r_0$, *where* $\lambda^{1/2} > 0$ *and the* C_n *are constants. Then* $\mu \equiv 0$ *in* Ω_{r_1}.

Proof. Applying the divergence theorem to the functions $\mu(X)$ and $\overline{[\mu(X)]}$, we have

$$\begin{aligned} 0 &= \int_{\Omega_{r_1}} (\mu(X)\,\Delta\overline{\mu(X)}\,dX - \overline{\mu(X)}\,\Delta\mu(X))\,dX \\ &= \int_{l_{r_1}} \left(\mu(r,\theta)\frac{\overline{\partial\mu(r,\theta)}}{\partial r} - \overline{\mu(r,\theta)}\frac{\partial\mu(r,\theta)}{\partial r}\right) r\,d\theta. \end{aligned}$$

Using the conditions (3.5) and (3.6), we have

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} |C_n(\lambda)|^2 H^{(1)}_{n/\alpha}(\lambda^{1/2}r_1)\frac{d}{dr}\overline{H^{(1)}_{n/\alpha}(\lambda^{1/2}r_1)} \\ &\quad - \sum_{n=1}^{\infty} |C_n(\lambda)|^2 \overline{H^{(1)}_{n/\alpha}(\lambda^{1/2}r_1)}\frac{d}{dr}H^{(1)}_{n/\alpha}(\lambda^{1/2}r_1). \end{aligned} \tag{3.7}$$

Now it follows from the theory of Bessel functions that

$$\begin{aligned} W\{J_{n/\alpha}(r), Y_{n/\alpha}(r)\} &= J_{n/\alpha}(r)\frac{dY_{n/\alpha}(r)}{dr} - Y_{n/\alpha}(r)\frac{d}{dr}J_{n/\alpha}(r) \\ &= 2/\pi r, \qquad n = 1, 2, \ldots, \end{aligned} \tag{3.8}$$

W denoting the Wronskian of the two solutions $J_{n/\alpha}$ and $Y_{n/\alpha}$. Combining (3.4), (3.7), and (3.8), we conclude that

$$0 = \sum_{n=1}^{\infty} |C_n(\lambda)|^2. \tag{3.9}$$

Thus $C_n(\lambda) = 0$, $n = 1, 2, \ldots$, and

$$\mu \equiv 0 \quad \text{in } \Omega_{(r_0 r_1)}. \tag{3.10}$$

Since μ satisfies the elliptic equation $(\Delta+\lambda)\mu = 0$ in Ω_{r_1}, (3.10) implies $\mu \equiv 0$ in Ω_{r_1}. Q.E.D.

Note that this is a stronger uniqueness theorem than that proven in I. We next show that subject to the conditions (3.5) and (3.6), (3.1) becomes a well-posed problem in χ. Let Λ_B denote a bounded subset in the upper half of the complex plane ($\text{Im}\,\lambda > 0$) such that Λ_B has positive distance from the origin. Suppose $\lambda_j \in \Lambda_B$, $j = 1, 2, \ldots$, and let $F_n(X)$ denote a sequence of $C^\infty(\overline{\Omega})$ functions which vanish in $\Omega - \Omega_{r_1}$, r_1 being independent of n. Set $\mu_n(X; \lambda_j) = (A - \lambda_j)^{-1} F_n(X)$.

THEOREM 3.2. (a) *Suppose* $\|F_n\|_{0(\Omega-\Omega_\varepsilon)} \leqq C$ *for each* $\varepsilon > 0$, $0 < \varepsilon < r_1 < \infty$. *Then* $\|\mu_n(\cdot\,; \lambda_j)\|_{0(\varepsilon,r)} \leqq C$, *the constant* C *depending only on* r *and* ε.

(b) *Suppose there exists a function* $F(X)$ *and a complex number* λ ($\text{Im}\,\lambda \geqq 0$) *such that* $\|F_n - F\|_{0(\Omega-\Omega_\varepsilon)} \to 0$ *and* $|\lambda_j - \lambda| \to 0$ *as* $n, j \to \infty$ *for each* $\varepsilon > 0$. *Then there* *exists a function* $\mu(X)$ *in* χ *such that* $\mu_n \to_\chi \mu$, $\bar{A}\mu = \lambda\mu + F$, *and* μ *satisfies conditions* (3.5) *and* (3.6).

The proof of this theorem is almost identical to that given in I and therefore the details will be omitted. The proof is indirect and may be outlined as follows. Under the assumption that part (a) of the theorem is false, we employ elliptic estimates, Rellich's selection theorem, and the Fourier expansion given by Lemma 3.1 to obtain a nonzero function $\mu(X)$ satisfying the hypothesis of Theorem 3.1. Theorem 3.1 then yields the required contradiction. Part (b) then follows from part (a) using similar arguments.

We next construct the generalized eigenfunctions, $W_n^+(X;\lambda)$, associated with the operator A. $W_n^+(X;\lambda)$ is to satisfy the boundary value problem: $(\Delta+\lambda)W_n^+(X;\lambda) = 0$ in Ω, $(\lambda>0)$, $W_n^+(X;\lambda)=0$ on $\dot{\Omega}$. We reduce this problem to that of solving (3.1) in the following manner. Set $V_n^+(X;\lambda) = W_n^+(X;\lambda) - W_n^0(X;\lambda)$. Thus $V_n^+(X;\lambda)$ must satisfy the boundary value problem:

$$(3.11)\qquad \begin{aligned} (\Delta+\lambda)V_n^+(X;\lambda) &= 0 \quad \text{in } \Omega, \\ V_n^+(X;\lambda) &= -W_n^0(X;\lambda) \quad \text{on } \dot{\Omega}. \end{aligned}$$

To reduce (3.11) to (3.1), let $\zeta(X)$ be a cutoff function satisfying the following conditions:

(1) $\zeta(X) \in C^\infty(\bar{\Omega})$,

(2) $\zeta(X) \equiv 0$ outside of some neighborhood of $\dot{\Omega} - \dot{S}$ and in some neighborhood of any singular point of Ω, and

(3) $\zeta(X) \equiv 1$ in some neighborhood of $\dot{\Omega} - \dot{S}$. Set

$$(3.12)\qquad \begin{aligned} G_n(X;\lambda) &= \zeta(X)W_n^0(X;\lambda), \\ F_n(X;\lambda) &= (-\Delta-\lambda)G_n(X;\lambda). \end{aligned}$$

By Theorems 3.1 and 3.2, there exists a unique solution $\mu_n^+(X;\lambda)$ in χ of the equation $\bar{A}\mu_n^+(X;\lambda) = \lambda\mu_n^+(X;\lambda) + F_n(X;\lambda)$, such that $\mu_n^+(X;\lambda)$ satisfies the radiation conditions (3.5) and (3.6). We now define

$$V_n^+(X;\lambda) = \mu_n^+(X;\lambda) - G_n(X;\lambda),$$

and

$$W_n^+(X;\lambda) = W_n^0(X;\lambda) + V_n^+(X;\lambda).$$

Note that $V_n^+(X;\lambda)$ satisfies the conditions (3.5) and (3.6).

Our aim is to establish the completeness and orthogonality of the functions $W_n^+(X;\lambda)$. For this purpose we shall make use of the following functions $V_n^+(X;\lambda;\kappa)$ as well as the properties of these functions given by Theorem 3.3. Suppose Im $\kappa>0$. Set

$$(3.13)\qquad F_n(X;\lambda;\kappa) = (-\Delta-\kappa)G_n(X;\lambda),$$

$$(3.14)\qquad \mu_n(X;\lambda;\kappa) = (A-\kappa)^{-1}F_n(X;\lambda;\kappa),$$

$$(3.15)\qquad V_n(X;\lambda;\kappa) = \mu_n(X;\lambda;\kappa) - G_n(X;\lambda),$$

and

$$W_n(X;\lambda;\kappa) = V_n(X;\lambda;\kappa)+W_n^0(X;\lambda). \tag{3.16}$$

For the sake of reference we state here without proof the following well-known result, which follows from the theory of elliptic partial differential equations, [1].

LEMMA 3.2. *Suppose B is a bounded domain in R^N such that $\dot{B}$ is a C^{2+j} surface, $j\geqq 0$. Furthermore, suppose the function $\mu(X)\in \dot{H}_1(B)$ and $\Delta\mu\in H_j(B)$. Then $\|\mu\|_{j+2(B)}\leqq C\|\Delta\mu\|_j$, the constant C being independent of μ.*

THEOREM 3.3. *Let ε be any positive number.* (a) *Suppose $\kappa\in\Lambda_B$. Then there exist constants C^ε such that* $\max_{\Omega(\varepsilon,r)}|V_n(X;\lambda;\kappa)|\leqq C^\varepsilon\lambda$. *$C^\varepsilon$ depends on $r>\varepsilon$ but is independent of λ in* $(0,\infty)$, $n=1,2,\dots$, *and κ in Λ_B.*

(b) *Suppose* $\operatorname{Im}\ \kappa>0$. *Then* $\|\mu_n(\cdot\,;\lambda;\kappa)\|_{0(\Omega)}\leqq C_\kappa\lambda$ *and* $\|A\mu_n(\cdot\,;\lambda;\kappa)\|_{0(\Omega)}\leqq C_\kappa\lambda$, *where the constant C_κ is independent of λ in* $(0,\infty)$ *and* $n=1,2,\dots$.

(c) *There exist constants d_n^ε such that* $\max_{\Omega_{\varepsilon,r}}|W_n(X;\lambda;\kappa)|\leqq d_n^\varepsilon$, *where d_n^ε depends on $r>\varepsilon$ but is independent of λ and κ for λ bounded and κ in Λ_B. Furthermore* $\sum_{n=1}^\infty d_n^\varepsilon<\infty$.

Proof. (a) This follows easily from Theorem 3.2, Lemma 3.2, the definition of $V_n(X;\lambda;\kappa)$, and Sobolev's inequality.

(b) The proof follows immediately from definitions (3.12)–(3.14), Lemma 3.2, and the fact that $(A-\kappa)^{-1}$ is a bounded operator.

(c) From definitions (3.12)–(3.16), we have

$$\begin{aligned} W_n(X;\lambda;\kappa) &= W_n^0(X;\lambda)-\zeta(r)W_n^0(X;\lambda)\\ &\quad +(A-\kappa)^{-1}[(-\Delta-\kappa)(\zeta(X)W_n^0(X;\lambda))]\\ &= I_1(X;\lambda)+I_2(X;\lambda)+I_3(X;\lambda;\kappa), \end{aligned} \tag{3.17}$$

and $W_n^0(X;\lambda)=(\pi\alpha)^{-1/2}J_{n/\alpha}(\lambda^{1/2}r)\sin(n\theta/\alpha)$. We now employ the following estimate for the Bessel functions $J_{n/\alpha}$, derived from Poisson's integral representation, [16]:

$$|J_\nu(Z)| \leqq |(\tfrac{1}{2}Z)^\nu K(Z)/\Gamma(\nu+1)|,$$

where $K(Z)$ is a continuous function of Z. Using the properties of the Gamma function $\Gamma(\nu)$, it follows readily that

$$\max_{\Omega_{\varepsilon,r}}|J_{n/\alpha}(\lambda^{1/2}r)| \leqq K(\lambda^{1/2}r)d_n^{\varepsilon(1)}, \tag{3.18}$$

where $\sum_{n=1}^\infty d_n^{\varepsilon(1)}<\infty$. Similarly, we have

$$|J'_{n/\alpha}(\lambda^{1/2}r)| \leqq K(\lambda^{1/2}r)d_n^{\varepsilon(2)},\quad \text{where } \sum_{n=1}^{\infty} d_n^{\varepsilon(2)} < \infty. \tag{3.19}$$

Estimate (3.18) immediately implies

$$\max_{\Omega_{\varepsilon,r}}|I_j(X;\lambda)| \leqq Cd_n^{\varepsilon(3)},\qquad j=1,2, \text{ where } d_n^{\varepsilon(3)} = \max(d_n^{\varepsilon(1)}, d_n^{\varepsilon(2)}). \tag{3.20}$$

We now consider $I_3(X; \lambda; \kappa)$. It follows from Theorem 3.2 and Lemma 3.2 that for $\delta > 0$

$$\begin{aligned}\max_{\Omega_{\varepsilon,r}} |(A-\kappa)^{-1}[(-\Delta-\kappa)(\zeta(r)W_n^0(X;\lambda))]| \\ \leqq C \max_{\Omega_{\varepsilon,r+\delta}} |(-\Delta-\kappa)[\zeta(r)W_n^0(X;\lambda)]|.\end{aligned} \tag{3.21}$$

Estimates (3.18), (3.19) and (3.21) yield

$$\max_{\Omega_{\varepsilon,r}} |I_3(X;\lambda;\kappa)| \leqq C d_n^{\varepsilon(4)}, \quad \text{where } \sum_{n=1}^{\infty} d_n^{\varepsilon(4)} < \infty. \tag{3.22}$$

The theorem now follows from (3.17), (3.20) and (3.22). Q.E.D.

4. **The expansion theorem.** We proceed to establish the completeness and orthogonality of the generalized eigenfunctions $W_n^+(X; \lambda)$. These results will follow from Theorems 4.1 and 4.2, respectively. First we define a pair of linear transformations from $C_0^\infty(\Omega)$ into H. Suppose $\phi(X) \in C_0^\infty(\Omega)$. For $\lambda > 0$ and $\operatorname{Im} \kappa > 0$, set

$$T^+\phi = \{\hat{\phi}_n^+(\lambda)\} = \left\{\int_\Omega \overline{W_n^+(X;\lambda)}\phi(X)\,dX\right\}, \tag{4.1}$$

and

$$\{\hat{\phi}_n(\lambda;\kappa)\} = \left\{\int_\Omega \overline{W_n(X;\lambda;\kappa)}\phi(X)\,dX\right\}. \tag{4.2}$$

LEMMA 4.1. *Suppose $\phi(X) \in C_0^\infty(\Omega)$. Then*

(a) *$\hat{\phi}_n(\lambda; \kappa)$ is continuous as a function of κ for fixed λ and n.*

(b) *$|\hat{\phi}_n(\lambda;\kappa)| \leqq C\lambda$, where C denotes a constant independent of λ, n, and κ for λ in $(0, \infty)$, $n = 1, 2, \ldots$, and κ in Λ_B.*

The proof is a simple consequence of the definition of $W_n(X; \lambda; \kappa)$, Theorem 3.2, Lemma 3.2 and the Schwarz inequality. We now state the main theorems of this section.

THEOREM 4.1. (a) *T^+ can be extended to a unitary transformation from $L_2(\Omega)$ onto some subspace $\mathscr{S}$ of H.*

(b) *For each $f(X)$ in $L_2(\Omega)$ and each bounded measurable function $s(\lambda)$:*

$$(s(A)f)\hat{}^{+} = \{s(\lambda)\hat{f}_n^+(\lambda)\}. \tag{4.3}$$

Furthermore $f \in D(A)$ if and only if $\{\lambda \hat{f}_n^+(\lambda)\} \in H$ and in this case $(Af)_n\hat{}^{+}(\lambda) = \lambda\hat{f}_n^+(\lambda)\}$.

(c) *For each $f(X)$ in $L_2(\Omega)$, we have*

$$f = \underset{a\downarrow 0,\, b\uparrow\infty}{\text{l.i.m.}} \sum_{n=1}^{\infty} \int_a^b \hat{f}_n^+(\lambda) W_n^+(X;\lambda)\,d\lambda. \tag{4.4}$$

THEOREM 4.2. *For each $g = \{g_n(\lambda)\}$ in H, we have*

$$g = T^+\left(\underset{a\downarrow 0,\, b\uparrow\infty}{\text{l.i.m.}} \sum_{n=1}^{\infty} \int_a^b g_n(\lambda) W_n^+(X;\lambda)\,d\lambda\right). \tag{4.5}$$

NOTE. Equation (4.4) is called the completeness relation and Equation (4.5) is called the orthogonality relation.

The proofs of these theorems are similar to the corresponding proofs in I and therefore only when changes occur will the arguments be explicitly given. Now suppose that $\{E_\lambda\}$ denotes the spectral resolution of the selfadjoint operator A. Thus $E_{\lambda^+}=E_\lambda$ and $A=\int \lambda\, dE_\lambda$. Our proof of Theorem 4.1 will be based on the well-known formula, [15]:

$$(4.6)\quad \tfrac{1}{2}((E_b+E_{b^-})f,f)_{0(\Omega)}-\tfrac{1}{2}((E_a+E_{a^-})f,f)_{0(\Omega)} = \lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b \|R_{\tau-i\varepsilon}f\|^2_{L_2(\Omega)}\,d\tau.$$

We shall evaluate the right-hand side of (4.6).

LEMMA 4.2. *Suppose* $\phi(X)\in C_0^\infty(\Omega)$ *and* Im $\kappa>0$. *Then*

$$(4.7)\quad \begin{aligned}\|R_{\bar\kappa}\phi\|^2_{L_2(\Omega)}-\int_{D^*}[R_{\bar\kappa}\phi(X)\cdot\overline{R_{\bar\kappa}\phi(X^*)}+R_{\bar\kappa}\phi(X^*)\cdot\overline{R_{\bar\kappa}\phi(X)}]\,dX \\ = \sum_{n=1}^{\infty}\int_0^\infty \frac{|\hat\phi_n(\lambda;\kappa)|^2\,d\lambda}{|\lambda-\kappa|^2}.\end{aligned}$$

Proof. Let $\mathscr{D}'$ represent the set of elements in H satisfying conditions (2.1) and (2.2) and denote by $\mathscr{D}$ the set of functions $\psi(X)$ defined in S such that $T^0\psi\in\mathscr{D}'$. For such a function $\psi(X)$ in $\mathscr{D}$, set

$$\begin{aligned}\psi^E(X) &= \psi(X) &&\text{for } X \text{ in } S\\ &= -\psi(X^*) &&\text{for } X \text{ in } D.\end{aligned}$$

We first establish the equation

$$(4.8)\qquad R_\kappa(\psi^E) = R(X;\kappa;\psi) = \sum_{n=1}^{\infty}\int_0^\infty \frac{\hat\psi_n^0(\lambda)W_n(X;\lambda;\kappa)\,d\lambda}{\lambda-\kappa}.$$

Since $W_n(X;\lambda;\kappa)=0$ on $\dot\Omega$, it follows from the definition of $R(X;\kappa;\psi)$, Theorem 3.3(b), and the properties of $\hat\psi_n^0(\lambda)$ that $R(X;\kappa;\psi)\in D(A)$. We now easily obtain

$$(-\Delta-\kappa)R(X;\kappa;\psi) = \sum_{n=1}^{\infty}\int_0^\infty \hat\psi_n^0(\lambda)W_n^0(X;\lambda)\,d\lambda = \psi^E(X),$$

since $W_n^0(X;\lambda)=-W_n^0(X^*;\lambda)$ for X in D. We have thus proven (4.8).

Since R_κ and $R_{\bar\kappa}$ are adjoints, it follows from (4.8) that

$$(4.9)\qquad (R_{\bar\kappa}\phi,\psi^E)_{(\Omega)} = (\phi, R_\kappa\psi^E)_{(\Omega)} = \sum_{n=1}^{\infty}\int_0^\infty \frac{\overline{\hat\psi_n^0}(\lambda)\hat\phi_n(\lambda;\kappa)\,d\lambda}{\lambda-\bar\kappa}$$

where we interchanged the order of the integrations and summation in the last step. Define $G_\kappa(X)=(R_\kappa\phi)(X^*)$ for X in D^*, and set $G_\kappa(X)\equiv 0$ for X in $S-D^*$. Since $\psi^E(X^*)=-\psi^E(X)$, we easily conclude from (4.9) that

$$(4.10)\qquad (\chi_S R_{\bar\kappa}\phi-G_\kappa(X),\psi)_{(S)} = \sum_{n=1}^{\infty}\int_0^\infty \overline{\hat\psi_n^0}(\lambda)\cdot(\lambda+n)^2\frac{\hat\phi_n(\lambda;\kappa)\,d\lambda}{(\lambda-\bar\kappa)(\lambda+n)^2}$$

where χ denotes the characteristic function.

Lemma 4.1 implies that the element

$$\left\{\frac{\hat{\phi}_n(\lambda;\kappa)}{(\lambda-\bar{\kappa})(\lambda+n)^2}\right\}\in H.$$

It follows from Theorem 2.1 that

$$(4.11)\qquad (\chi_S R_{\bar{\kappa}}\phi-G_\kappa,\psi)_{(S)}=\sum_{n=1}^{\infty}\int_0^\infty \frac{(\chi_S R_{\bar{\kappa}}\phi-G_\kappa)_n^{\hat{0}}(\lambda)}{(\lambda+n)^2}\,\overline{\hat{\psi}_n^0(\lambda)}(\lambda+n)^2\,d\lambda.$$

It is easily seen that the set of elements $\{\hat{\psi}_n^0(\lambda)(\lambda+n)^2\}$, where $\psi\in\mathscr{D}$, is dense in H. We thus conclude from (4.10) and (4.11) that

$$(4.12)\qquad (\chi_S R_{\bar{\kappa}}\phi-G_\kappa)_n^{\hat{0}}(\lambda)=\hat{\phi}_n(\lambda;\kappa)/(\lambda-\bar{\kappa}).$$

Again employing Theorem 2.1, we have proven the lemma. Q.E.D.

We are now ready to prove the following key result.

THEOREM 4.3. *For each $\phi(x)$ in $C_0^\infty(\Omega)$:*

$$(4.13)\qquad \tfrac{1}{2}((E_b+E_{b^-})\phi,\phi)_{0(\Omega)}-\tfrac{1}{2}((E_a+E_{a^-})\phi,\phi)_{0(\Omega)}=\sum_{n=1}^{\infty}\int_a^b|\hat{\phi}_n^+(\lambda)|^2\,d\lambda.$$

Proof. Suppose $\kappa=\tau+i\varepsilon$. By Lemma 4.2, we have

$$(4.14)\qquad \begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b &\|R_{\tau-i\varepsilon}\phi\|^2_{L_2(\Omega)}\,d\tau\\ &-\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b\int_{D^*}[R_{\bar{\kappa}}\phi(X)\cdot\overline{R_{\bar{\kappa}}\phi(X^*)}+R_{\bar{\kappa}}\phi(X^*)\overline{R_{\bar{\kappa}}\phi(X)}]\,dX\,d\tau\\ &=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b\sum_{n=1}^{\infty}\int_0^\infty\frac{|\hat{\phi}_n(\lambda;\kappa)|^2\,d\lambda\,d\tau}{|\lambda-\kappa|^2}\\ &=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\sum_{n=1}^{\infty}\int_0^\infty\int_a^b\frac{|\hat{\phi}_n(\lambda;\kappa)|^2\,d\tau\,d\lambda}{(\lambda-\tau)^2+\varepsilon^2},\end{aligned}$$

using Fubini's theorem to interchange the order of the integrations and summation. Before completing the proof of the theorem we must establish the following lemma.

LEMMA 4.3. *Suppose $\phi(X)\in C_0^\infty(\Omega)$. Then*

$$(4.15)\qquad \lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\sum_{n=1}^{\infty}\int_{b+1}^\infty\int_a^b\frac{|\hat{\phi}_n(\lambda;\kappa)|^2\,d\tau\,d\lambda}{(\lambda-\tau)^2+\varepsilon^2}=0.$$

Proof. Suppose $\psi(X)\in\mathscr{D}$. It may be shown as in I that

$$(4.16)\qquad \left|\sum_{n=1}^{\infty}\int_0^\infty\hat{\psi}_n^0(\lambda)\overline{[\hat{\phi}_n(\lambda;\kappa)-\hat{\phi}_n^{0^E}(\lambda)]}\,d\lambda\right|^2\le C\int_B|U_\psi(X;\kappa)|^2\,dX,$$

where B=support of $\phi(X)$, C is a constant depending only on $\phi(X)$, $\hat{\phi}_n^{0^E}(\lambda)=\int_\Omega\phi(X)W_n^0(X;\lambda)\,dX$, and

$$U_\psi(X;\kappa)=\sum_{n=1}^{\infty}\int_0^\infty\hat{\psi}_n^0(\lambda)V_n^+(X;\lambda;\kappa)\,d\lambda.$$

We wish to bound the integral occurring on the right-hand side of (4.16). Note that $U_\psi(X;\kappa)$ satisfies the boundary value problem:

$$(4.17)\qquad \begin{aligned}(-\Delta-\kappa)U_\psi &= 0 && \text{in } \Omega,\\ U_\psi(X;\kappa) &= -\psi^E(X) && \text{on } \dot{\Omega}.\end{aligned}$$

It follows easily that

$$U_\psi(X) = -\zeta(X)\psi^E(X)+(A-\kappa)^{-1}(-\Delta-\kappa)\zeta\psi^E(X).$$

We observe that Theorem 3.2(a) and the linearity of A imply:

$$\|(A-\kappa)^{-1}F\|_{0(B)} \leqq C\|F\|,$$

the constant C being independent of $\kappa=\tau+i\varepsilon$ ($\tau\in[a,b]$, $\varepsilon_0\geqq\varepsilon>0$), as well as all functions $F(X)$ in $C^\infty(\Omega)$ with the same bounded support. We now conclude easily that

$$\int_B |U_\psi(X;\kappa)|^2\,dX \leqq C\int_\Omega |(-\Delta-\kappa)\psi^E(X)|^2\,dX.$$

Combining this with estimate (4.16) and Theorem 2.1, we obtain

$$(4.18)\qquad \begin{aligned}&\left|\sum_{n=1}^\infty \int_0^\infty \hat{\psi}_n^0(\lambda)\overline{[\hat{\phi}_n^{0E}(\lambda)-\hat{\phi}_n(\lambda;\kappa)]}\,d\lambda\right|^2\\ &\qquad \leqq C\|(-\Delta+1)\psi^E\|^2_{0(\Omega)} \leqq C\sum_{n=1}^\infty\int_0^\infty (1+\lambda)^2|\hat{\psi}_n^0(\lambda)|^2\,d\lambda,\end{aligned}$$

where C depends only on $\phi(X)$.

Expressing estimate (4.18) in the form

$$(4.19)\qquad \left|\sum_{n=1}^\infty\int_0^\infty \frac{Q_n(\lambda)\overline{[\hat{\phi}_n^{0E}(\lambda)-\hat{\phi}_n(\lambda;\kappa)]}\,d\lambda}{1+\lambda}\right|^2 \leqq C\sum_{n=1}^\infty\int_0^\infty |Q_n(\lambda)|^2\,d\lambda,$$

where $Q_n(\lambda)=\hat{\psi}_n^0(\lambda)(1+\lambda)$, we see that (4.19) holds on all of H since it holds on a dense subset. It follows from Lemma 4.2 that the element

$$\left\{\frac{\hat{\phi}_n^{0E}(\lambda)-\hat{\phi}_n(\lambda;\kappa)}{1+\lambda}\right\}\in H.$$

Therefore an application of (4.19) to this element yields

$$\sum_{n=1}^\infty\int_0^\infty \frac{|\hat{\phi}_n^{0E}(\lambda)-\hat{\phi}_n(\lambda;\kappa)|^2\,d\lambda}{(1+\lambda)^2}\leqq C.$$

Since C is independent of κ, Lemma 4.3 follows immediately. Q.E.D.

We now return to the proof of Theorem 4.3. By (4.14) and (4.15), we have

$$(4.20)\qquad \begin{aligned}&\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b \|R_{\tau-i\varepsilon}\phi\|^2_{L_2(\Omega)}\,d\tau\\ &\qquad -\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b\int_{D^*}[R_{\bar{\kappa}}\phi(X)\cdot\overline{R_{\bar{\kappa}}\phi(X^*)}+R_{\bar{\kappa}}\phi(X^*)\cdot\overline{R_{\bar{\kappa}}\phi(X)}]\,dX\,d\tau\\ &\qquad = \lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\sum_{n=1}^\infty\int_0^{b+1}\int_a^b\frac{|\hat{\phi}_n(\lambda;\kappa)|^2}{(\lambda-\tau)^2+\varepsilon^2}\,d\tau\,d\lambda.\end{aligned}$$

First note that the second term on the left-hand side is zero because

$$\left|\frac{\varepsilon}{\pi}\int_a^b d\tau \int_{D^*} [R_{\bar\kappa}\phi(X)\overline{R_{\bar\kappa}\phi(X^*)}+R_{\bar\kappa}\phi(X^*)\overline{R_{\bar\kappa}\phi(X)}]\, dX\right| \leq \frac{\varepsilon}{\pi} C \max_{X\in D^*;\, \tau\in[a,b]} |R_{\bar\kappa}\phi(X)|\, |R_{\bar\kappa}\phi(X^*)| \leq \varepsilon C,$$

where the constant C is independent of ε. This follows from Theorem 3.2, Lemma 3.2, and Sobolev's inequality.

It follows from Theorem 3.3(c) that the series on the right-hand side of (4.20) converges uniformly with respect to ε. Hence we may interchange the limit and the summation to obtain

$$\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b \|R_{\tau-i\varepsilon}\phi\|^2_{L_2(\Omega)}\, d\tau = \sum_{n=1}^{\infty}\int_0^{b+1}\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b \frac{|\hat\phi_n(\lambda;\kappa)|^2\, d\tau\, d\lambda}{(\lambda-\tau)^2+\varepsilon^2}. \tag{4.21}$$

We next employ the following result [14]:

If $f(\tau, \varepsilon)$ is a continuous function of τ and ε for $0\leqq\varepsilon\leqq\varepsilon_0$ and $\alpha\leqq\tau\leqq\beta$, then

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_\alpha^\beta \frac{f(\tau,\varepsilon)}{(A-\tau)^2+\varepsilon^2} &= 0 \qquad \text{if } A<\alpha \text{ or } \beta<A,\\ &= f(A,0) \quad \text{if } \alpha<A<\beta.\end{aligned} \tag{4.22}$$

(4.21) and (4.22) now yield

$$\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_a^b \|R_{\tau-i\varepsilon}\phi\|^2_{L_2(\Omega)}\, d\tau = \sum_{n=1}^{\infty}\int_a^b |\hat\phi_n^+(\lambda)|^2\, d\lambda. \tag{4.23}$$

Combining (4.6) and (4.23), we have proven the theorem. Q.E.D.

Having established Theorem 4.3, we may prove Theorems 4.1 and 4.2 exactly as in I (since A has a continuous spectrum by Remark 3.1). The proofs will be omitted here. Theorems 4.1 and 4.2 together imply that A is unitarily equivalent to A_0. The functions $W_n^+(X;\lambda)$ form a complete, orthonormal set of generalized eigenfunctions of A. The conditions (3.5) and (3.6) satisfied by $V_n^+(X;\lambda)$ are called "outgoing" radiation conditions. In the same way, we may construct another complete, orthonormal set of generalized eigenfunctions, $W_n^-(X;\lambda)=W_n^0(X;\lambda)+V_n^-(X;\lambda)$, where $V_n^-(X;\lambda)$ satisfies certain "incoming" radiation conditions at infinity. These conditions are the same as (3.5) and (3.6) with $H^{(1)}_{n/\alpha}(\lambda^{1/2}r)$ replaced by $H^{(2)}_{n/\alpha}(\lambda^{1/2}r)=J_{n/\alpha}(\lambda^{1/2}r)-iY_{n/\alpha}(\lambda^{1/2}r)$. For $\phi(X)$ in $C_0^\infty(\Omega)$, set

$$T^-\phi = \{\hat\phi_n^-(\lambda)\} = \left\{\int_\Omega \phi(X)\overline{W_n^-(X;\lambda)}\, dX\right\}.$$

Theorems 4.1 and 4.2 now hold with T^+ replaced by T^-.

5. **Invariance of the wave operators.** In this section we employ methods similar to those used in II to establish the existence, unitarity, and invariance of the wave operators $W^\pm=W^\pm(\phi(A),\phi(A_0))$ for a class of functions $\phi(\lambda)$ satisfying the following conditions:

(i) $\phi(\lambda)$ is a real-valued function defined on $[0,\infty)$.

(ii) There exists a sequence of subintervals $I_K=(\lambda_{K-1}, \lambda_K)$ of $(0,\infty)$ such that $(0,\infty)=\bigcup_{K=-\infty}^{\infty} I_K$, $\lambda_K\to 0$ as $K\to -\infty$, $\lambda_K\to\infty$ as $K\to\infty$, $\phi(\lambda)\in C^3(I_K)$, and $\phi'(\lambda)\neq 0$ for λ in I_K.

We define the wave operators as follows. For each $f(X)$ in $L_2(S)$ $(L_2(\Omega))$, set $U_0(t)f=e^{it\phi(A_0)}f$ $(U(t)f=e^{it\phi(A)}f)$, $-\infty<t<\infty$. Recall that we are assuming throughout that $\Omega\supset S$. The arguments may easily be modified in the more general case. We let $j(X)$ denote a cutoff function defined in Ω such that:

(1) $j(X)\equiv 0$ for $r_0\geqq r$, (2) $j(X)\equiv 1$ for $r\geqq r'>r_0$, and (3) $j(X)\in C^\infty(\bar{S})$. For $\psi(X)$ in $C_0^\infty(S)$, set $J\psi(X)=j(X)\psi(X)$ for X in S and zero for X in $\Omega-S$, and extend J by continuity as a mapping from $L_2(S)$ into $L_2(\Omega)$. We define

$$W^{\pm}f=\lim_{t\to\pm\infty} U(-t)JU_0(t)f=\lim_{t\to\pm\infty} W(t)f \tag{5.1}$$

for $f(X)$ in $L_2(S)$. We now prove the main result of this section.

THEOREM 5.1. (a) *Suppose* $f\in L_2(S)$. *Then* $W^{\pm}f=\lim_{t\to\pm\infty} W(t)f$ *exists in* $L_2(\Omega)$.

(b) *If* $\phi'(\lambda)>0$ $(\phi'(\lambda)<0)$ *on* I_K, *then* $(W^{\pm}f)_n^{\hat{\pm}}(\lambda)=\hat{f}_n^0(\lambda)((W^{\pm}f)_n^{\hat{\mp}}(\lambda)=\hat{f}_n^0(\lambda))$ *for* λ *in* I_K.

Proof. Consider the subset $\mathscr{D}$ of elements $f(X)$ in $L_2(S)$ satisfying the conditions:

(a) $\hat{f}_n^0(\lambda)\in C_0^\infty(0,\infty)$, $n=1,2,\ldots$

(b) $\hat{f}_n^0(\lambda)\equiv 0$ for $n\geqq M$, and

(c) the support of $\hat{f}_n^0(\lambda)\subset I_K$, where $\phi'(\lambda)>0$ in I_K.

To begin with we assume that $\phi(\lambda)\in C^\infty(0,\infty)$. Applying the same arguments as in the proof of Lemma 4.1 in II, we obtain

$$(W(T)f)_n^{\hat{\pm}}(\lambda) = (jf)_n^{\hat{\pm}}(\lambda)-\int_0^T dt\int_\Omega H(X;\lambda;t;0)(\Delta+\lambda)(j(X)W_n^{\pm}(X;\lambda))\,dX \tag{5.2}$$

where

$$\begin{aligned} H(X;\lambda;t;\varepsilon) &= i\sum_{j=1}^{\infty}\int_0^\infty W_j^0(X;\zeta)\,\frac{(\phi(\zeta)-\phi(\lambda))}{\zeta-\lambda}\,(\zeta-\lambda)\hat{f}_j^0(\zeta)\\ &\quad\cdot\exp\left(it\left[\phi(\zeta)-\phi(\lambda)+i\varepsilon\left(\frac{\phi(\zeta)-\phi(\lambda)}{\zeta-\lambda}\right)\right]\right)=d\zeta \qquad \text{for } X \text{ in } S,\\ &= 0 \qquad \text{for } X \text{ in } \Omega-S. \end{aligned} \tag{5.3}$$

Now suppose $\phi(\lambda)$ is an arbitrary function satisfying (i), (ii) and suppose a closed subinterval $[a,b]$ of I_K contains the support of $\hat{f}_n^0(\lambda)$ for all n. Let $\phi_j(\lambda)$ denote a sequence of C^∞ functions satisfying (i), (ii) and such that $\phi_j(\lambda)\to\phi(\lambda)$ and $\phi_j'(\lambda)\to\phi'(\lambda)$ uniformly on $[a,b]$. It follows readily that $H^j(X;\lambda;t;0)\to H(X;\lambda;t;0)$ and $(W^j(t)f)_n^{\hat{\pm}}(\lambda)\to(W(t)f)_n^{\hat{\pm}}(\lambda)$ for all λ in $[a,b]$ where H^j and W^j are defined by (5.1) and (5.3) respectively with $\phi(\lambda)$ replaced by $\phi_j(\lambda)$. Thus (5.2) holds for λ in I_K. Similarly (5.2) may be shown to hold for $\lambda>0$ in I_j, $j=1,2,\ldots$.

Again it follows as in the proof of Lemma 4.1 in II that as $t \to \pm\infty$:

$$(W(t)f)_n^{\hat{\pm}}(\lambda) \to \hat{f}_n^0(\lambda)$$

uniformly in an arbitrary interval $[c, d] \subset I_j$, $j = 1, 2, \ldots$. We may easily conclude from this that

$$(W(t)f)^{\hat{\pm}} \rightharpoonup \hat{f}^0 \qquad \text{as } t \to \pm\infty, \tag{5.4}$$

$\rightharpoonup$ denoting weak convergence in H. We next show that

$$\|(W(t)f)^{\hat{\pm}}\|_H \to \|\hat{f}^0\|_H \quad \text{as } t \to \pm\infty. \tag{5.5}$$

Since $U_0(T)$ and $U(t)$ are isometric, we have

$$\begin{aligned} \big|\, \|W(t)f\|_{L_2(\Omega)} - \|f\|_{L_2(S)} \big| &= \big|\, \|JU_0(t)f\|_{L_2(\Omega)} - \|U_0(t)f\|_{L_2(S)} \big| \\ &\leqq \|(1-J)U_0(t)f\|_{L_2(S)} \\ &\leqq C \max_B |U_0(t)f(X)| \end{aligned}$$

where B is a bounded subset of S. We may deduce from the unperturbed expansion theorem and integration by parts that

$$\max_B |U_0(t)f(X)| = O((1+|t|)^{-1}).$$

Thus (5.5) is proven.

We now employ the following well-known theorem of functional analysis. Suppose $\{Y_n\}$ denotes a weakly convergent sequence of elements in a Hilbert space. Suppose also that $\|Y_n\| \to \|Y\|$, Y being the weak limit of Y_n. Then Y_n converges strongly to Y. In our case the hypotheses of this theorem hold as a result of (5.4) and (5.5). Thus

$$(W(t)f)^{\hat{\pm}} \to \hat{f}^0 \qquad \text{in } H \text{ as } t \to \pm\infty. \tag{5.6}$$

Finally, suppose $f = f_1 + \cdots + f_l$, where f_j satisfies conditions (a)–(c) for $1 \leqq j \leqq l$ and suppose that the functions f_j have their supports contained in mutually disjoint intervals, I_j. Then we see from (5.6) that $(W(t)f_j)^{\hat{\pm}} \to \hat{f}_j^0$ as $t \to \pm\infty$. By the linearity of $W(t)$, T^0, and $T^{\pm}$, we see that the conclusions of the theorem hold for these functions $f(X)$. Similar arguments suffice if $\phi'(\lambda) < 0$ in some of the intervals I_j. Since the set of such functions is dense in $L_2(S)$, the theorem is proven. Q.E.D.

Having established the existence and unitarity of the wave operators $W^{\pm}$, we define the scattering operator $\mathscr{S}$ as follows. For each function $f(X)$ in $L_2(S)$, set

$$\mathscr{S}f = (W^+)^{-1}W^-f. \tag{5.7}$$

It follows from Theorem 5.1 that $\mathscr{S}$ is a unitary mapping from $L_2(S)$ onto $L_2(S)$. Finally we remark that this theory could easily be carried over to the case of the wave equation in Ω. The Hilbert space employed in this case is the space of initial data in Ω with the corresponding energy norm. Since the details are essentially the same as those in II, they will be omitted.

6. **The scattering matrix.** We shall obtain a more suitable expression for the scattering operator $\mathcal{S}$ defined by equation (5.7). The dependence of $\mathcal{S}$ on the incoming solutions $V_n^-(X;\lambda)$ will show up clearly in this expression. Let $\mathcal{D}$ be the dense subset of $L_2(S)$ consisting of functions $f(X)$ such that $(T^-)^{-1}T^0 f(X) \in C_0^\infty(\Omega)$.

THEOREM 6.1. *Suppose $f\in\mathcal{D}$ and $\lambda\in I_\kappa$, where $\phi'(\lambda)>0$ on I_κ. Then*

$$(\mathcal{S}f)_n^{\hat{0}}(\lambda) = \hat{f}_n^0(\lambda)+\sum_{n'=1}^{\infty} t_{n,n'}(\lambda)\hat{f}_{n'}^0(\lambda),$$

where

$$t_{n,n'}(\lambda) = -2\pi i\int_\Omega \overline{F_n(X;\lambda)}V_{n'}^-(X;\lambda)\,dX,$$

and $F_n(X;\lambda)$ is defined by (3.12).

Note that by (3.12), $F_n(X;\lambda)=(-\Delta-\lambda)\zeta(X)W_n^0(X;\lambda)$, where the cutoff function $\zeta(X)$ was defined in §3. We denote by r_1 a real number chosen such that $\zeta(X)\equiv 0$ for $r>r_1>r_0$. We make the following additional assumption for the function $\zeta(X)$. $\partial^j\zeta(r,\theta)/\partial\theta^j=0$ on $\dot{\Omega}$, $j=1,2,\ldots$.

Proof. It follows from Theorem 5.1 that

$$(6.1)\quad \begin{aligned}(\mathcal{S}f)_n^{\hat{0}}(\lambda) &= (T^+(T^-)^{-1}\hat{f}^0)_n(\lambda)\\ &= \hat{f}_n^0(\lambda)+\int_\Omega \overline{(V_n^+(X;\lambda)-V_n^-(X;\lambda))}((T^-)^{-1}\hat{f}^0)(X)\,dX.\end{aligned}$$

Using the definition of $V_n^\pm(X;\lambda)$ and Theorem 4.1, we have from (6.1);

$$(6.2)\quad \begin{aligned}(\mathcal{S}f)_n^{\hat{0}}(\lambda) = \hat{f}_n^0(\lambda)+\int_\Omega \lim_{\varepsilon\downarrow 0}\sum_{n'=1}^{\infty}\int_0^\infty &\left[\frac{\overline{(F_n(\cdot\,;\lambda))_{n'}^{\hat{-}}(\lambda')}}{\lambda'-\lambda+i\varepsilon}-\frac{\overline{(F_n(\cdot\,;\lambda))_{n'}^{\hat{-}}(\lambda')}}{\lambda'-\lambda-i\varepsilon}\right]\\ &\cdot\overline{W_{n'}^-(X;\lambda')}(T^-)^{-1}\hat{f}^0(X)\,d\lambda'\,dX.\end{aligned}$$

Since $(T^-)^{-1}\hat{f}^0$ has compact support, we may use Theorems 3.3, 4.1 and 4.2 and the fact that $F_n(X;\lambda)\in D(A^j)$, $j=1,2,\ldots$, to conclude that

$$(6.3)\quad \begin{aligned}(\mathcal{S}f)_n^{\hat{0}}(\lambda) = \hat{f}_n^0(\lambda)+\lim_{\varepsilon\downarrow 0}\sum_{n=1}^{\infty}\int_0^\infty &\overline{(F_n(\cdot\,;\lambda))_{n'}^{\hat{-}}(\lambda')}\hat{f}_{n'}^0(\lambda')\\ &\cdot\left[\frac{1}{\lambda'-\lambda+i\varepsilon}-\frac{1}{\lambda'-\lambda-i\varepsilon}\right]d\lambda'.\end{aligned}$$

Since $F_n(X;\lambda)$ and $f(X)\in D(A^j)$, $j=1,2,\ldots$, we obtain with the aid of (4.22):

$$(6.4)\quad (\mathcal{S}f)_n^{\hat{0}}(\lambda) = \hat{f}_n^0(\lambda)+\sum_{n'=1}^{\infty} t_{n,n'}(\lambda)\hat{f}_{n'}^0(\lambda)$$

where

$$t_{n,n'}(\lambda) = -2\pi i \int_\Omega \overline{F_n(X;\lambda)}(W^0_{n'}(X;\lambda)+V^-_{n'}(X;\lambda))\,dX.$$

$$\begin{aligned}\int_\Omega \overline{F_n(X;\lambda)}W^0_{n'}(X;\lambda)\,dX &= \int_{l_{r_0}} \Bigg[-\overline{W^0_n(X;\lambda)}\frac{\partial W^0_{n'}(X;\lambda)}{\partial r} \\ &\qquad\qquad + \frac{\partial \overline{W^0_n(X;\lambda)}}{\partial r} W^0_{n'}(X;\lambda)\Bigg]\,d\theta \\ &= 0.\end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5) we have proven the theorem. Q.E.D.

It appears from Theorem 6.1 that the coefficients $t_{n,n'}(\lambda)$ depend on the function $\zeta(X)$ as well as $W_{n'}(X;\lambda)$ and $W^0_n(X;\lambda)$. This apparent dependence on $\zeta(X)$ is only superficial, however, as will be seen from the following corollary.

COROLLARY 6.1. $t_{n,n'}(\lambda)=4(\pi\alpha)^{1/2}C_{n,n'}(\lambda)$ *where*

$$V_{n'}(X;\lambda) = \sum_{j=1}^\infty C_{j,n'}(\lambda)H_{j/\alpha}(\lambda^{1/2}r)\sin\frac{j}{\alpha}\theta \qquad \textit{for } r \geqq r_2.$$

Proof. We have from Theorem 6.1,

$$\begin{aligned}t_{n,n'}(\lambda) &= -2\pi i \int_\Omega \overline{F_n(X;\lambda)}V^-_{n'}(X;\lambda)\,dx \\ &= -2\pi i \int_\Omega V^-_{n'}(X;\lambda)(-\Delta-\lambda)(\zeta(X)W^0_n(X;\lambda))\,dx \\ &= -2\pi i \int_\Omega W_{n'}(X;\lambda)(-\Delta-\lambda)((\zeta(X)-1)W^0_n(X;\lambda))\,dx,\end{aligned}$$

since $(-\Delta-\lambda)W^0_n(X;\lambda)=0$ in Ω.

Employing the divergence theorem, equation (3.8), and the definitions of $\zeta(X)$, $W^0_n(X;\lambda)$, and $V^-_{n'}(X;\lambda)$ we obtain

$$\begin{aligned}t_{n,n'}(\lambda) &= -2\pi i \int_{\Omega_{r_1}\cup(\Omega-\Omega_{r_1})} (\zeta(X)-1)W^0_n(X;\lambda)(-\Delta-\lambda)W^-_{n'}(X;\lambda)\,d\lambda \\ &\quad -2\pi i\Bigg(\int_{l_{r_1}} (\zeta(X)-1)W^0_n(X;\lambda)\frac{\partial}{\partial r}W^-_{n'}(X;\lambda) \\ &\qquad\qquad - W^-_{n'}(X;\lambda)\frac{\partial}{\partial r}(\zeta(X)-1)W^0_n(X;\lambda)\Bigg)r_1\,d\theta \\ &= -4(\pi\alpha)^{1/2}C_{n,n'}(\lambda).\end{aligned}$$

Q.E.D.

Note that since

$$\overline{H^{(1)}_{n/\alpha}(\lambda^{1/2}r)} = H^{(2)1/2}_{n/\alpha}(\lambda r),$$

and $W^0_n(X;\lambda)$ is real, it follows from Theorem 3.1 that

$$\overline{V^+_n(X;\lambda)} = V^-_n(X;\lambda). \tag{6.6}$$

Thus Theorem 6.1 also expresses the scattering operator in terms of the outgoing solutions. If $\phi'(\lambda)<0$ on I_κ, analogous formulas hold for the elements of the scattering matrix.

7. **Extensions of previous results.** We next extend the previous results in various directions. The methods we have used are rather general and depend on the fact that the operator A_0 may be treated by separation of variables. Thus we may obtain similar results concerning boundary perturbations of various other domains (of arbitrary dimension) as well as other selfadjoint elliptic operators for which separation of variables may be employed. For example, the results of I and II now follow even for the case in which the perturbed cylinder Ω is not a subdomain of the unperturbed cylinder S.

Instead of boundary perturbations, we may treat perturbations in the coefficients of the equation or in the boundary conditions. For instance, we have the following two theorems, (where A_0 and S again denote the operator and domain defined in §2).

THEOREM 7.1. *Suppose A denotes the selfadjoint elliptic operator given by $-\sum_{i,j=1,\ldots,N}(\partial/\partial X_i)(a_{ij}\partial/\partial X_j)-q(X)$ acting on functions defined in S and vanishing on $\dot{S}$. Suppose that $\delta_{ij}-a_{ij}(X)\equiv 0$ and $q(X)\equiv 0$ for r sufficiently large and that $a_{ij}(X)\in C^1(\bar{S})$, $q(X)\in C(\bar{S})$, (δ_{ij} denoting the Kronecker delta). Then two sets of generalized eigenfunctions $W_n^{\pm}(X;\lambda)$ may be constructed satisfying the radiation condition* (3.5), (3.6). *Furthermore the results of Theorems* 4.1, 4.2, 5.1 *and* 6.1 *hold in this case with A replaced by A^C (the continuous part of A).*

THEOREM 7.2. *Suppose the selfadjoint operator A is given by $-\Delta$ acting on functions $\mu(X)$ defined in S and satisfying the boundary conditions: $\sigma(X)\partial u/\partial\nu=\mu$ on $\dot{S}$, where ν denotes the outward directed normal from S, $\sigma(X)\in C^2(\bar{S})$ and $\sigma(X)$ vanishes for r sufficiently large. Then the results of Theorem* 7.1 *hold.*

The task of constructing the generalized eigenfunctions $W_n^{\pm}(X;\lambda)$ corresponding to Theorem 7.1 is reduced to that of solving the boundary value problem:

$$\sum_{i,j=1,\ldots,N}\frac{\partial}{\partial X_i}\left(a_{ij}(X)\frac{\partial}{\partial X_j}V_n(X;\lambda)\right)+\lambda V_n(X;\lambda)$$

$$= \sum_{i,j=1,\ldots,N}\frac{\partial}{\partial X_i}\left((\delta_{ij}-a_{ij}(X))\frac{\partial W_n^0(X;\lambda)}{\partial X_j}\right)-q(X)W_n^0(X;\lambda)\quad \text{in } S, \tag{7.1}$$

$$V_n(X;\lambda)=0\quad\text{on } \dot{S}.$$

For Theorem 7.2, the boundary value problem becomes

$$(\Delta+\lambda)V_n(X;\lambda)=0\quad\text{in } S,$$

$$\sigma(X)\frac{\partial V_n(X;\lambda)}{\partial\nu}-V_n(X;\lambda) = -\sigma(X)\frac{\partial W_n^0(X;\lambda)}{\partial\nu}\quad\text{on } \dot{S}. \tag{7.2}$$

Both (7.1) and (7.2) may be solved by the methods employed in §3. The solutions satisfy the same radiation conditions as before. The results of §§4–6 then follow in exactly the same way. Note that these results also hold when A_0 corresponds to the boundary condition

$$\partial\mu/\partial n = \sigma(\theta)\mu \quad \text{on } \dot{S}, \quad \text{where } \sigma \leqq 0.$$

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